Postulates of quantum mechanics

State space

- Postulate 1:
  - Any isolated physical system is associated with a complex vector space with inner product (Hilbert space) = "*state space*"
  - System is described by a unit vector.
- *Qubit*
  - Two-dimensional vector space
  - Orthonormal basis $|0\rangle$, $|1\rangle$
  - Arbitrary state vector $|\psi\rangle=\alpha|0\rangle + \beta|1\rangle$
  - Normalization condition: $\langle\psi|\psi\rangle = |\alpha|^2+|\beta|^2 = 1$

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Time-evolution

- Postulate 2:
  - The time-evolution of a *closed* system is described by a *unitary transformation.*

    $|\psi(t_2)\rangle = U(t_2,t_1)|\psi(t_1)\rangle$
  - in *continuous* time
  - Schrödinger equation

    $i\hbar\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = H|\psi(t)\rangle,\qquad U(t_2,t_1) = \exp[-\tfrac{i}{\hbar}H(t_2-t_1)]$
  - $H = \sum_E E|E\rangle\langle E|$ (spectral decomposition)
  - $|E\rangle \to \exp(-iEt/\hbar)|E\rangle$

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Quantum measurement

- Postulate 3:
  - Quantum measurements are described by *measurement operators* $\{M_m\}$
    - *m:* measurement outcomes
  - Probability that result *m* occurs

    $p(m) = \langle\psi|M_m^+M_m|\psi\rangle$
  - State after measurement $\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^+M_m|\psi\rangle}}$
  - *Completeness* of measurement operators

    $\sum_m M_m^+M_m = I \Rightarrow \sum_m\langle\psi|M_m^+M_m|\psi\rangle = \sum_m p(m) = 1$

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Example: measurement of a qubit in computational basis

- Measurement of state: $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$
- Measurement operators: $M_0=|0\rangle\langle 0|$, $M_1=|1\rangle\langle 1|$
  - Hermitian: $M_0^\dagger=M_0$, $M_1^\dagger=M_1$
  - $M_0^2=M_0$, $M_1^2=M_1$
  - Completeness: $M_0^\dagger M_0+M_1^\dagger M_1=M_0+M_1=1$
- Probability that meas. outcome is 0

  $p_0 = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle=|\alpha|^2$
- States after measurement

  $\frac{M_0|\psi\rangle}{|\alpha|} = \frac{\alpha}{|\alpha|}|0\rangle \sim |0\rangle,\quad \frac{M_1|\psi\rangle}{|\beta|} = \frac{\beta}{|\beta|}|1\rangle \sim |1\rangle$

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Distinguishability of quantum states

- Distinguish orthonormal states: $|\psi_i\rangle (i=1,\cdots,n)$
  - Define meas. Operators

    $M_i \equiv |\psi_i\rangle\langle\psi_i|$

    $M_0 \equiv \sqrt{I-\sum_i|\psi_i\rangle\langle\psi_i|}$ $\Rightarrow$ Completeness $\sum_i M_i^+M_i + M_0^+M_0 = I$
  - When the state $|\psi_i\rangle$ is prepared, meas. result =1 with certainty

    $p(i) = \langle\psi_i|M_i|\psi_i\rangle = 1$

  ⇒ Possible to distinguish the orthonormal states.
- Non-orthonormal states $\langle\psi_j|\psi_i\rangle \neq 0$ $(i\neq j)$

  $\langle\psi_j|M_i|\psi_j\rangle \neq 0$

  ⇒ Impossible to distinguish the non-orthonormal states

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Projective (von Neumann) measurements

- A special case of Postulate 3
- A projective measurement
  - Measurement operators: $P_m = |m\rangle\langle m|$
    - Projector onto the eigenstate of M with eigenvalue m
  - Corresponding *observable* $M = \sum_m m P_m$
  - Probability that meas. outcome is m

    $p(m) = \langle\psi|P_m|\psi\rangle$
- States after measurement

  $$\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$$

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Projective (von Neumann) measurements II

- Completeness $\sum_m P_m^\dagger P_m = \sum_m P_m = 1$
- Additional property: Orthogonality

  $P_m^\dagger P_{m'} = P_m P_{m'} = \delta_{m,m'} M_m$
- Easy to calculate
  - average values for projective measurement

    $$E(M) = \sum_m m p(m) = \sum_m \langle\psi|P_m|\psi\rangle$$
    $$= \langle\psi|\left(\sum_m P_m\right)|\psi\rangle = \langle\psi|M|\psi\rangle$$
- "Measurement in a basis $|m\rangle$"

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Example: Projective measurement on single qubits

- Observable: Z

  Eigenvalues: ±1
  - Eigenvectors: $|0\rangle$, $|1\rangle$
  - Measurement of Z on state

    $$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
    - Result = 0 with prob. $\langle\psi|0\rangle\langle 0|\psi\rangle$= ½
    - Result = 1 with prob. $\langle\psi|1\rangle\langle 1|\psi\rangle$= ½
- Observable: $v\cdot\sigma \equiv v_1X + v_2Y + v_3Z$

  (measurement of spin along v-axis)

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POVM measurements

- <u>P</u>ositive <u>O</u>perator-<u>V</u>alued <u>M</u>easure
  - Measurement operators $\{M_m\}$
  - Probability that result m occurs

    $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$
  - POVM element $E_m \equiv M_m^\dagger M_m$

    (positive operator)

    $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \geq 0, \ \sum_m E_m = I$
- Example: projective measurement

  $E_m \equiv P_m^\dagger P_m = P_m$

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Example: POVM

- Distinguish two states?

  $$|\psi_1\rangle = |0\rangle,\ |\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
- POVM: (Positive, $\sum_m E_m = 1$)

  $$E_1 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|,$$
  $$E_2 \equiv \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}, \quad \Rightarrow \quad \langle\psi_1|E_1|\psi_1\rangle = 0,\ \langle\psi_2|E_2|\psi_2\rangle = 0$$
  $$E_3 \equiv 1 - E_1 - E_2$$
- Measurement
  - Result = $E_1 \Rightarrow |\psi_2\rangle$
  - Result = $E_2 \Rightarrow |\psi_1\rangle$
  - Result = $E_3 \Rightarrow$ No answer

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Phase

- Global phase
  - $e^{i\theta}|\psi\rangle$
  - Equals to $|\psi\rangle$ up to global phase
  - No difference in measurement

    $\langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|e^{-i\theta}M_m^\dagger M_m e^{i\theta}|\psi\rangle$
- Relative phase

  $$\frac{|0\rangle + |1\rangle}{\sqrt{2}},\ \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

  ⇒ Physically observable differences

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Composite systems

- Postulate 4:
  - State space of a composite system = tensor product of state spaces comprising physical systems

$$|\psi_1\rangle\otimes|\psi_2\rangle\otimes\cdots\otimes|\psi_n\rangle$$

- Product state

$$(\alpha_1|0\rangle_1+\beta_1|1\rangle_1)\otimes(\alpha_2|0\rangle_2+\beta_2|1\rangle_2)$$

- Entangled state ≠ product state

$$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$$

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No cloning theorem

No-cloning theorem

- *Unknown quantum state can not be copied.*
- Proof: Try to copy an unknown state $|\psi\rangle$ into $|s\rangle$
  - Initial state: $|\psi\rangle\times|s\rangle$
  - Unitary operator U (copying) for two states, $|\psi\rangle$ and $|\phi\rangle$

$$U:|\psi\rangle\otimes|s\rangle\rightarrow U(|\psi\rangle\otimes|s\rangle)=|\psi\rangle\otimes|\psi\rangle$$
$$U:|\phi\rangle\otimes|s\rangle\rightarrow U(|\phi\rangle\otimes|s\rangle)=|\phi\rangle\otimes|\phi\rangle$$

$$\langle\psi|\phi\rangle=(\langle\psi|\phi\rangle)^2\Rightarrow\langle\psi|\phi\rangle=0,\ \text{or } 1$$

⇒ Only orthogonal states can be copied. No general quantum cloning possible.

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Formal description of errors

Qubit and environment

- Unitary evolution of a qubit and environment
  - Qubit in $|0\rangle$ or $|1\rangle$

$$|0\rangle\otimes|0\rangle_E\rightarrow|0\rangle\otimes|e_{00}\rangle_E+|1\rangle\otimes|e_{01}\rangle_E$$
$$|1\rangle\otimes|0\rangle_E\rightarrow|0\rangle\otimes|e_{10}\rangle_E+|1\rangle\otimes|e_{11}\rangle_E$$

  - An arbitrary state: $|\psi\rangle=a|0\rangle+b|1\rangle$

$$(a|0\rangle+b|1\rangle)\otimes|0\rangle_E\rightarrow a(|0\rangle\otimes|e_{00}\rangle_E+|1\rangle\otimes|e_{01}\rangle_E)+b(|0\rangle\otimes|e_{10}\rangle_E+|1\rangle\otimes|e_{11}\rangle_E)$$
$$\equiv|\psi\rangle\otimes|e_I\rangle_E+(X|\psi\rangle)\otimes|e_I\rangle_E+(Y|\psi\rangle)\otimes|e_Y\rangle_E+(Z|\psi\rangle)\otimes|e_Z\rangle_E$$

    - What happens to the qubit?
      - Nothing (I): $I|\psi\rangle=a|0\rangle+b|1\rangle$
      - Bit flip (X): $X|\psi\rangle=a|1\rangle+b|0\rangle$
      - Phase flip (Z): $Z|\psi\rangle=a|0\rangle-b|1\rangle$
      - Both ($Y=iXZ$): $Y|\psi\rangle=a|1\rangle-b|0\rangle$
    - Environment

$$|e_I\rangle_E=\tfrac{1}{2}(|e_{00}\rangle_E+|e_{01}\rangle_E),\ |e_X\rangle_E=\tfrac{1}{2}(|e_{01}\rangle_E+|e_{10}\rangle_E),$$
$$|e_Y\rangle_E=\tfrac{1}{2}(|e_{01}\rangle_E-|e_{10}\rangle_E),\ |e_Z\rangle_E=\tfrac{1}{2}(|e_{00}\rangle_E-|e_{01}\rangle_E)$$

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Errors on *n*-qubits

- Error on 1 qubit
  - 2×2 *unitary matrix*: *I, X, Z, Y*
- Error on *n* qubits
  - $2^n\times2^n$ *unitary matrix*: $\{I, X, Z, Y\}^{\otimes n}\equiv\{E_a\}$
  - Unitary evolution of the qubits and environment: $|\psi\rangle\otimes|0\rangle_E\rightarrow\sum_a E_a|\psi\rangle\otimes|e_a\rangle_E$
- What's error correction?
  - Correct subset of $\{E_a\}$: $\varepsilon\subseteq\{E_a\}$
  - Procedure
    - Perform collective measurement
    - Diagnose which error $\varepsilon\in E_a$ occurred
    - Correct the error by applying $E_a^\dagger=E_a$

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Conditions for error correction

- Necessary condition
  - ${}_L\langle j|E_b^\dagger E_a|i\rangle_L=0,\ i\neq j$
    $\{|i\rangle_L\}$: orthonormal basis for the code subspace
- Sufficient condition
  - ${}_L\langle j|E_b^\dagger E_a|i\rangle_L=\delta_{a,b}\delta_{i,j}$
    (non-degenerate code)
- Necessary and sufficient condition
  - ${}_L\langle j|E_b^\dagger E_a|i\rangle_L = C_{a,b}\delta_{i,j}$
    - $C_{a,b} = {}_L\langle i|E_b^\dagger E_a|i\rangle_L$: independent of i

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Assumptions for error correction and fault-tolerant quantum computation

- Basic assumptions
  - Constant error rate
  - Weakly correlated errors
  - Parallel operation
  - Reusable memory
- Additional assumptions
  - Fast measurements
  - Fast and accurate classical processing
  - No leakage
  - Non-local quantum gates

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