Grover's algorithm

School on Quantum Computing @Yagami
Day 1, Lesson 4
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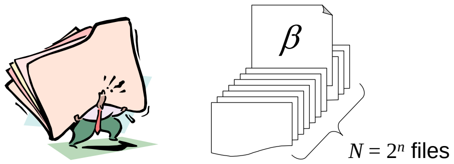
Database searching

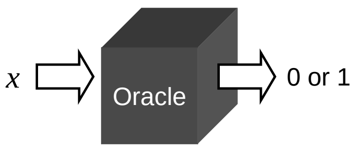
Find the desired file indexed as "β"
among $N = 2^n$ files



Oracle

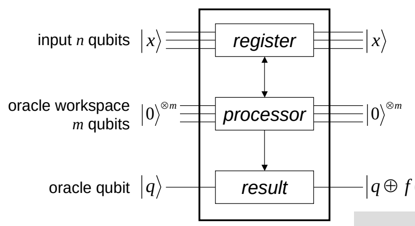
A black box that can ***recognize*** the solution, whose internal working is represented by a binary function $f(x)$

$$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$



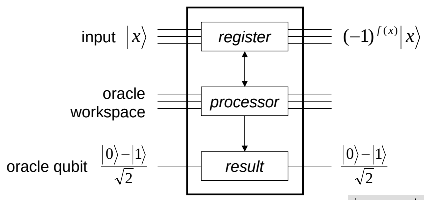
Oracle

$$|x\rangle|q\rangle \xrightarrow{O} |x\rangle|q \oplus f(x)\rangle$$



$$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$$

Oracle



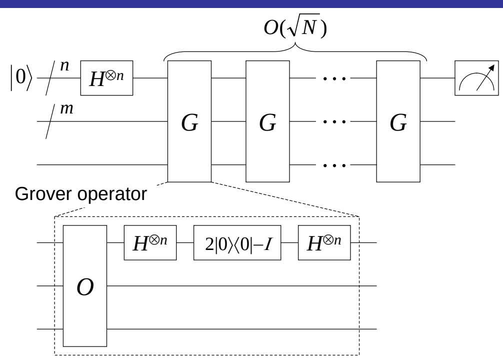
$$|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \xrightarrow{O} (-1)^{f(x)}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

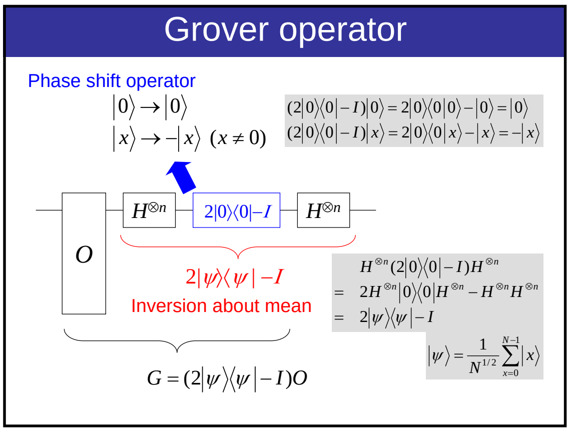
$$|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle = \begin{cases} |0\rangle - |1\rangle & \text{if } f(x) = 0 \\ |1\rangle - |0\rangle & \text{if } f(x) = 1 \end{cases} = (-1)^{f(x)}(|0\rangle - |1\rangle)$$

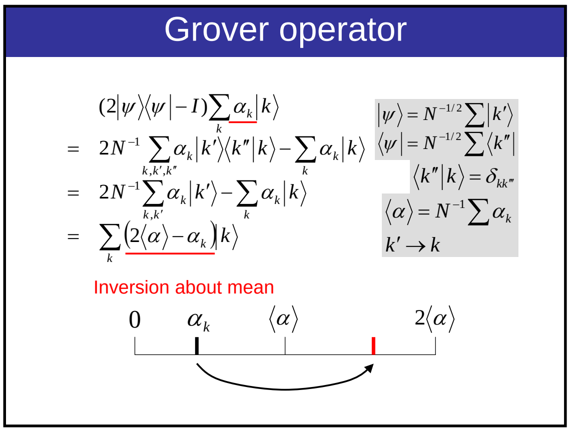
Simplify

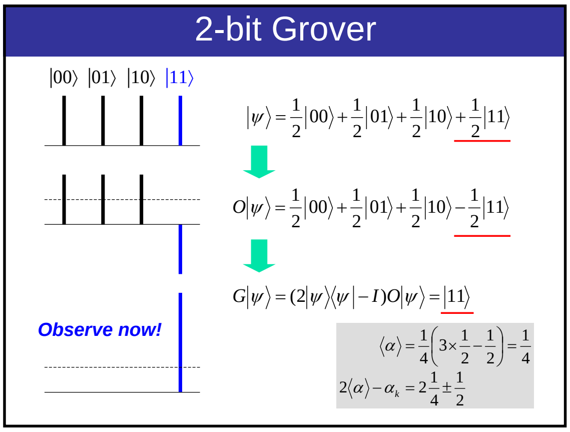
$$|x\rangle \xrightarrow{O} (-1)^{f(x)}|x\rangle$$

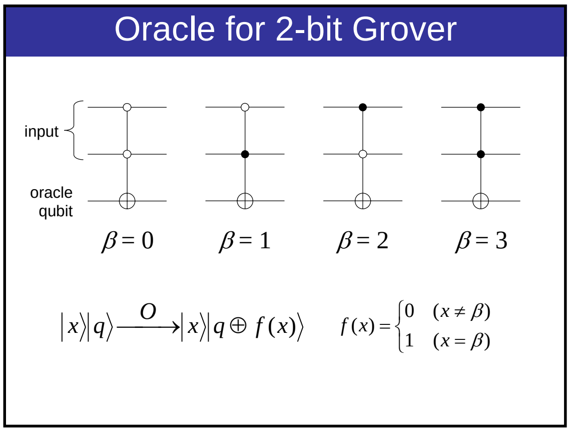
Quantum search algorithm

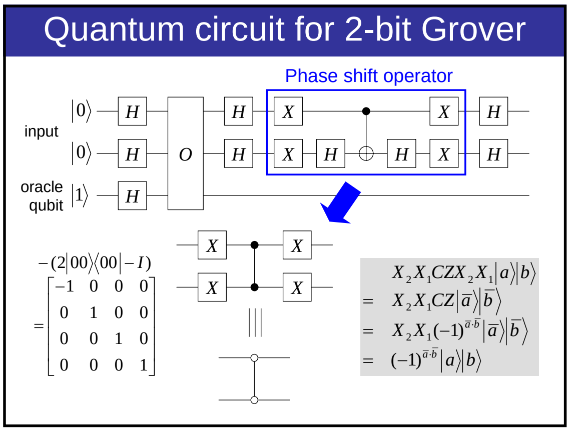


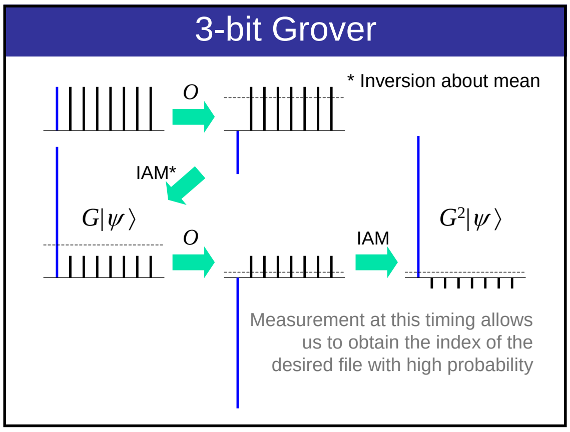

Grover operator
Phase shift operator
$|0\rangle \to |0\rangle$
$|x\rangle \to -|x\rangle \ (x \neq 0)$
$(2|0\rangle\langle 0| - I)|0\rangle = 2|0\rangle\langle 0|0\rangle - |0\rangle = |0\rangle$
$(2|0\rangle\langle 0| - I)|x\rangle = 2|0\rangle\langle 0|x\rangle - |x\rangle = -|x\rangle$
O
$H^{\otimes n}$
$2|0\rangle\langle 0| - I$
$H^{\otimes n}$
$2|\psi\rangle\langle\psi| - I$
Inversion about mean
$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}$
$= 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n} - H^{\otimes n}H^{\otimes n}$
$= 2|\psi\rangle\langle\psi| - I$
$|\psi\rangle = \frac{1}{N^{1/2}}\sum_{x=0}^{N-1}|x\rangle$
$G = (2|\psi\rangle\langle\psi| - I)O$


Grover operator
$(2|\psi\rangle\langle\psi| - I)\sum_k \alpha_k|k\rangle$
$= 2N^{-1}\sum_{k,k',k''}\alpha_k|k'\rangle\langle k''|k\rangle - \sum_k \alpha_k|k\rangle$
$= 2N^{-1}\sum_{k,k'}\alpha_k|k'\rangle - \sum_k \alpha_k|k\rangle$
$= \sum_k (2\langle\alpha\rangle - \alpha_k)|k\rangle$
$|\psi\rangle = N^{-1/2}\sum|k'\rangle$
$\langle\psi| = N^{-1/2}\sum\langle k''|$
$\langle k''|k\rangle = \delta_{kk''}$
$\langle\alpha\rangle = N^{-1}\sum\alpha_k$
$k' \to k$
Inversion about mean
0
α_k
$\langle\alpha\rangle$
$2\langle\alpha\rangle$


2-bit Grover
$|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$
$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$
$O|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
$G|\psi\rangle = (2|\psi\rangle\langle\psi| - I)O|\psi\rangle = |11\rangle$
Observe now!
$\langle\alpha\rangle = \frac{1}{4}\left(3\times\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{4}$
$2\langle\alpha\rangle - \alpha_k = 2\frac{1}{4} \pm \frac{1}{2}$


Oracle for 2-bit Grover
input
oracle qubit
$\beta = 0$
$\beta = 1$
$\beta = 2$
$\beta = 3$
$|x\rangle|q\rangle \xrightarrow{O} |x\rangle|q \oplus f(x)\rangle$
$f(x) = \begin{cases} 0 & (x \neq \beta) \\ 1 & (x = \beta) \end{cases}$


Quantum circuit for 2-bit Grover
Phase shift operator
input
oracle qubit
$|0\rangle$
$|0\rangle$
$|1\rangle$
H
O
X
$-(2|00\rangle\langle 00| - I)$
$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$X_2X_1CZX_2X_1|a\rangle|b\rangle$
$= X_2X_1CZ|\bar{a}\rangle|\bar{b}\rangle$
$= X_2X_1(-1)^{\bar{a}\cdot\bar{b}}|\bar{a}\rangle|\bar{b}\rangle$
$= (-1)^{\bar{a}\cdot\bar{b}}|a\rangle|b\rangle$


3-bit Grover
* Inversion about mean
O
IAM*
$G|\psi\rangle$
O
IAM
$G^2|\psi\rangle$
Measurement at this timing allows us to obtain the index of the desired file with high probability

Geometric visualization

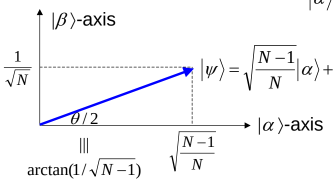
Sum over all x except β

$$|\alpha\rangle \equiv \frac{1}{\sqrt{N-1}}\sum_x{}' |x\rangle$$

$$\langle\alpha|\alpha\rangle = \frac{1}{N-1}\sum_{x,x'}{}'\langle x'|x\rangle = 1, \quad \langle\alpha|\beta\rangle = 0$$

The initial state $|\psi\rangle$ is visualized as a vector in the real 2D plane spanned by $|\alpha\rangle$ and $|\beta\rangle$

$$|\alpha\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \quad |\beta\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$$



$$|\psi\rangle = \sqrt{\frac{N-1}{N}}|\alpha\rangle + \frac{1}{\sqrt{N}}|\beta\rangle = \begin{pmatrix}\cos[\theta/2]\\ \sin[\theta/2]\end{pmatrix}$$

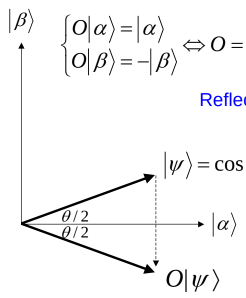
$\frac{1}{\sqrt{N}}$, $\sqrt{\frac{N-1}{N}}$, $\theta/2 = \arctan(1/\sqrt{N-1})$

Geometric visualization

$$\begin{cases} O|\alpha\rangle = |\alpha\rangle \\ O|\beta\rangle = -|\beta\rangle \end{cases} \Leftrightarrow O = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$|x\rangle \xrightarrow{O} (-1)^{f(x)}|x\rangle, \quad f(x) = \begin{cases}0 & (x \neq \beta)\\ 1 & (x = \beta)\end{cases}$$

Reflection about the $|\alpha\rangle$-axis



$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$

Geometric visualization

$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$

$$\begin{aligned}(2|\psi\rangle\langle\psi| - I)|\alpha\rangle &= 2\cos\frac{\theta}{2}|\psi\rangle - |\alpha\rangle \\ &= \left(2\cos^2\frac{\theta}{2} - 1\right)|\alpha\rangle + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|\beta\rangle \\ &= \cos\theta|\alpha\rangle + \sin\theta|\beta\rangle\end{aligned}$$

$$\begin{aligned}(2|\psi\rangle\langle\psi| - I)|\beta\rangle &= 2\sin\frac{\theta}{2}|\psi\rangle - |\beta\rangle \\ &= 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|\alpha\rangle + \left(2\sin^2\frac{\theta}{2} - 1\right)|\beta\rangle \\ &= \sin\theta|\alpha\rangle - \cos\theta|\beta\rangle\end{aligned}$$

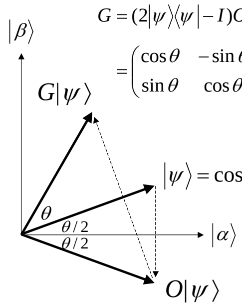
$$\Rightarrow \quad 2|\psi\rangle\langle\psi| - I = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

Reflection about the vector $|\psi\rangle$

Geometric visualization

$$G = (2|\psi\rangle\langle\psi| - I)O = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

The product of two reflections is a rotation

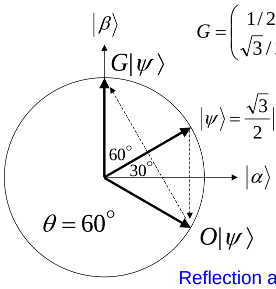


$$|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$$

2-bit Grover

Rotation by 60 degrees

$$G = \begin{pmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}$$

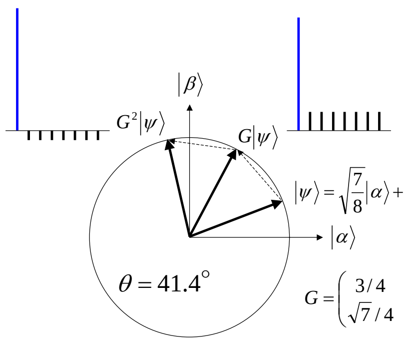


$$|\psi\rangle = \frac{\sqrt{3}}{2}|\alpha\rangle + \frac{1}{2}|\beta\rangle$$

$$O = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Reflection about the $|\alpha\rangle$-axis

3-bit Grover



$$|\psi\rangle = \sqrt{\frac{7}{8}}|\alpha\rangle + \frac{1}{\sqrt{8}}|\beta\rangle$$

$$G = \begin{pmatrix}3/4 & -\sqrt{7}/4\\ \sqrt{7}/4 & 3/4\end{pmatrix}$$

Performance

The state after repeating the Grover iteration k times

$$G^k|\psi\rangle = \cos\left(\frac{2k+1}{2}\theta\right)|\alpha\rangle + \sin\left(\frac{2k+1}{2}\theta\right)|\beta\rangle$$

We terminate the iteration when

$$\frac{2k'+1}{2}\theta \approx \frac{\pi}{2}$$

Assume θ is small (i.e., N large), then

$$\sin\frac{\theta}{2} = \frac{1}{\sqrt{N}} \approx \frac{\theta}{2}$$

The number of steps required to find the desired file

$$k' \approx \frac{\pi}{4}(\sqrt{N}-1) = O(\sqrt{N})$$

Optimality

- Classical algorithms take $O(N)$ operations for searching N items
- Grover's algorithm can search N items by calling the oracle only $O(N^{1/2})$ times
- Is is shown that Grover's algorithm is optimal, i.e., any quantum algorithms require at least $O(N^{1/2})$ times oracle callings for searching
- The proof is beyond the scope of this introductory lecture

Quiz

Complete the quantum circuit for 3-bit Grover to search the file #0