Topological code

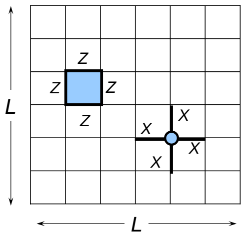
Toric code
- Generalized CSS code

Toric code

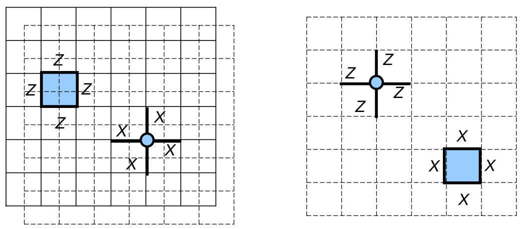
- 2D surface on a torus ($L \times L$)
- Qubits are on the links
  - $2L^2$ total qubits
- Stabilizer generators

$$Z_P = \prod_{l \in P} Z_l,\quad X_S = \prod_{l \in S} X_l$$

  - L^2 plaquette operators Z_P
  - L^2 site operators X_S
  - Product of all X_S/ Z_P - operators = Identity
  - → $2L^2$-2 independent operators
  - → 2 encoded qubits



Dual lattice

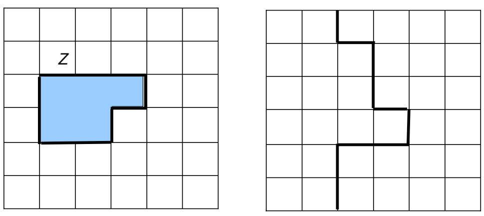


Dual lattice
Z ↔ X
Plaquette ↔ Site

Cycles (Z)

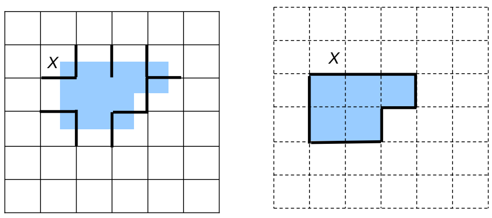
Both commute with stabilizer generators

- Trivial cycle = a product of stabilizer generators Z_P
- → does nothing on the code subspace
- Non-trivial cycle
- → acts non-trivially on the code subspace
- = logic operation



Trivial dual cycle (X)

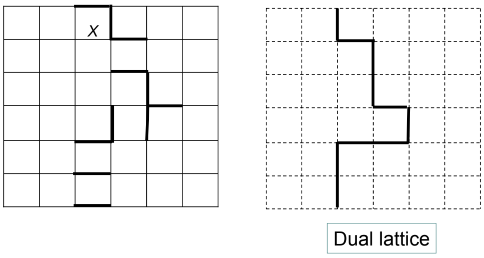
- Trivial cycle = product of stabilizer generators X_S
- → does nothing on the code subspace



Dual lattice

Non-trivial dual cycle (X)

- Non-trivial cycle
- → acts non-trivially on the code subspace
- = logic operation



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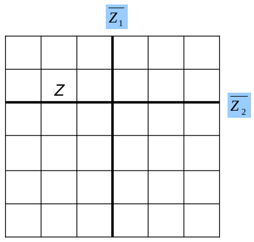
Pauli operators Z on encoded qubits

- Two fundamental non-trivial cycles of Z's of the torus → $\bar{Z}_1, \bar{Z}_2$



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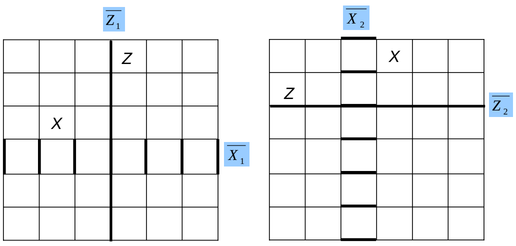
Pauli operators X on encoded qubits

- Two fundamental non-trivial cycles of Z's
  → $\bar{Z}_1, \bar{Z}_2$
- Anti-commuting cycles of X's → $\bar{X}_1, \bar{X}_2$



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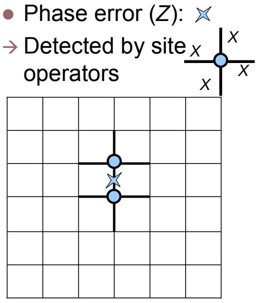
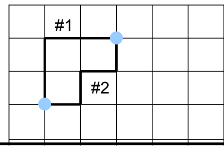
Error detection – site defects

- Phase error (Z):
- → Detected by site operators



- Error syndrome is ambiguous
  - Error chains #1 and #2 give the same syndrome
    - → Trivial cycle: code undamaged
    - → Non-trivial cycle: damaged



Error syndrome = site "defects"

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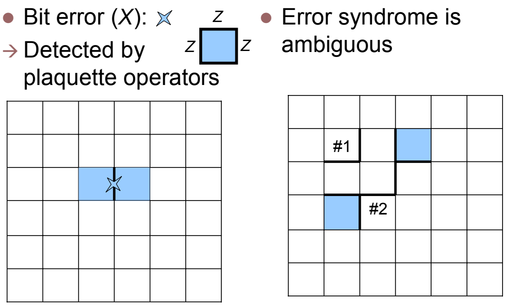
Error detection – plaquette defects

- Bit error (X):
- → Detected by plaquette operators
- Error syndrome is ambiguous



Error syndrome = plaquette "defects"

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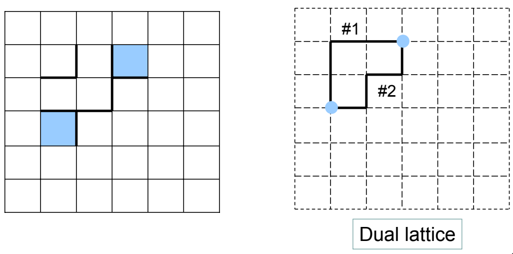
X error chains in dual lattice

- Error chains from the previous slide
- In dual lattice



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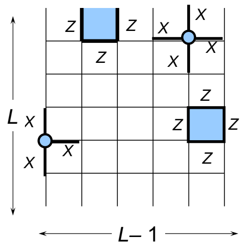
Planar code

Planar code

- 2D surface with *boundaries* $L\times(L-1)$
- Qubits are on the links
  - $L^2 + (L+1)^2$ total qubits
- Stabilizer generators

$$Z_P = \prod_{l\in P} Z_l, \quad X_S = \prod_{l\in S} X_l$$

  - $L(L-1)$ plaquette operators
  - $L(L-1)$ site operators
  - No constraint due to the boundary conditions
  - $L^2 + (L+1)^2 - 2L(L-1)^2$
  - → 1 encoded qubit



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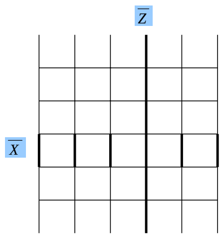
Pauli operators on encoded qubits

- Non-trivial cycles of Z's → $\overline{Z}$
- Anti-commuting cycles of X's → $\overline{X}$



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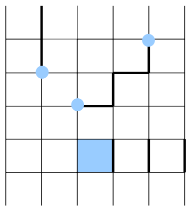
Defects due to errors

- Site and plaquette defects may appear singularly
  - In toric code, defects appear only in pairs



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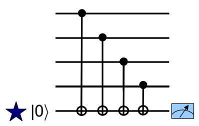
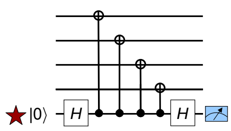
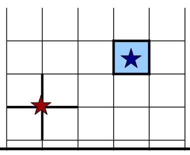
Circuit for syndrome measurement

- Plaquette operator $Z_P = ZZZZ$
- Site operator $X_S = XXXX$







No need for Shor or Steane state for ancilla for fault-tolerance
← at most two errors

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Issues of error correcting codes

- Better error correcting codes require longer block length size
  - Longer CSS code or;
  - Concatenated code
  - → Error correction requires operations involving increasing number of qubits entangled non-locally
- Topological codes
  - Only local operations (among four neighboring spins)
  - Avoid overhead imposed by non-locality

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