Fault tolerant quantum computation

Quantum error correcting code:

The 7-qubit (Steane) code

Fault tolerance

- Assumptions used so far:
  - No errors when storing qubits
  - No errors when processing qubits
- In reality:
  - Logic operations may be faulty (encoding, syndrome measurement, etc)
  - Errors might propagate during logic operations
- What we need:
  - Syndrome measurement; and
  - a universal set of quantum gates on encoded qubits without unacceptable error propagation
  - A code that can correct more than one error in the code block

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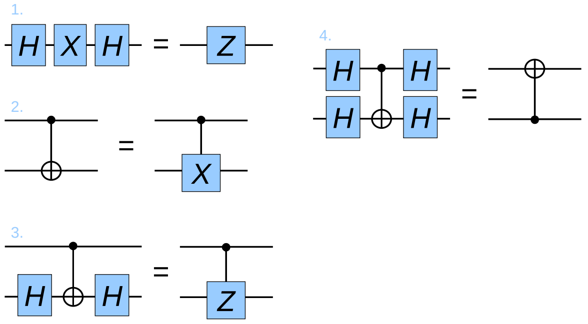
Useful identities



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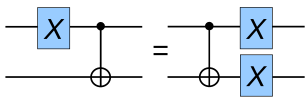
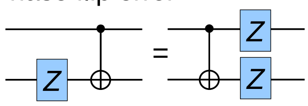
Propagation of errors

- Bit flip error



- Phase flip error



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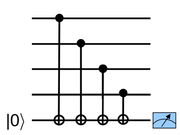
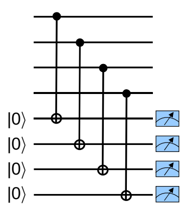
Syndrome measurement

- Bad
  - Errors propagate and accumulate!



- Good
  - No error propagation



  - Too much information is copied
  - Measurement destroys encoded qubits

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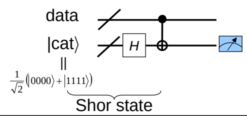
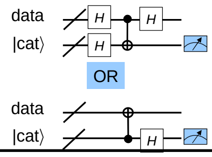
Preparing ancillary qubits

- Shor state $|\text{Shor}\rangle = \frac{1}{\sqrt{8}} \sum_{\text{even } w} |w\rangle$
  - Four qubits: equal weighted superposition state of all even strings
  - Reveal error syndrome only
  - 24 ancillary qubits (in 6 Shor states) and 24 CNOT gates

Bit flip error detection



Phase flip error detection



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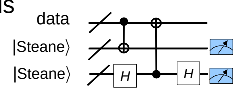
Preparing ancillary qubits II

- Steane state
  - 7 qubits: equal weighted superposition state of all Hamming codewords

$|\mathrm{Steane}\rangle = \frac{1}{4}\sum_{w\in \mathrm{Hamming}}|w\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}\rangle + |\bar{1}\rangle\right)$



  - Procedures:
    - CNOT gate between corresponding data qubits and ancillary qubits
    - Apply check matrix to the ancillary qubits
  - 14 ancillary qubits (in 2 Steane states) and 14 CNOT gates

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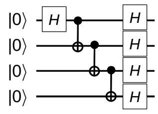
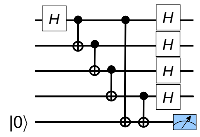
Verification of ancillary qubits

- Preparation of Shor State (not fault-tolerant)



- Ancillary qubits may have errors!
  - Phase flip errors propagates to the data qubits.
  - Bit flip errors do not damage the data qubits
- Verification



  - If the ancillary qubit is 1: One or more bit flip errors (before *H*) occurred
    → Discard
  - Measurement may be faulty
    → Repeat

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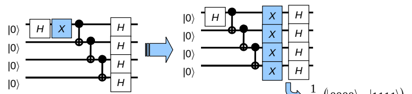
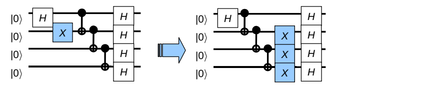
Error propagation in Shor state

- Bit flip errors
  - Bit flip on qubit #1 (no error)



$\frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$

  - Bit flip on qubit #2



$\frac{1}{\sqrt{2}}\left(|0111\rangle + |1000\rangle\right)$

  - Bit flip on qubit #3 $\frac{1}{\sqrt{2}}\left(|0011\rangle + |1100\rangle\right)$
  - Bit flip on qubit #4 $\frac{1}{\sqrt{2}}\left(|0001\rangle + |1110\rangle\right)$

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Fault-tolerant quantum gates

- Logic gates on encoded qubits
  = bitwise application
  - $\bar{Z} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7$
  - $\bar{X} = X_1X_2X_3X_4X_5X_6X_7$
  - Hadamard gate $\bar{H}$ interchanges $\bar{Z}$ and $\bar{X}$
    $\bar{H} = H_1H_2H_3H_4H_5H_6H_7$
  - Phase gate *S* $\quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
    - Bitwise application of *ZS*
  - CNOT gate
    - Bitwise application of CNOT

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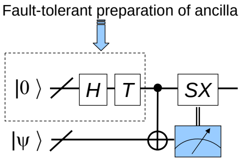
Fault-tolerant π/8 gate

Fault-tolerant preparation of ancilla Θ

$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} R_z\left(\frac{\pi}{4}\right)$



$\Theta = \frac{|0\rangle + e^{i\pi/4}|1\rangle}{\sqrt{2}}$

$|\Theta\rangle\left(a|0\rangle + b|1\rangle\right) \xrightarrow{\mathrm{CNOT}} \frac{1}{\sqrt{2}}\left[|0\rangle\left(a|0\rangle + b|1\rangle\right) + e^{i\pi/4}|1\rangle\left(a|1\rangle + b|0\rangle\right)\right]$

$= \frac{1}{\sqrt{2}}\left[\left(a|0\rangle + e^{i\pi/4}b|1\rangle\right)|0\rangle + \left(b|0\rangle + e^{i\pi/4}a|1\rangle\right)|1\rangle\right]$

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

If measurement result is 1, apply SX
→ $T|\psi\rangle$ (up to global phase)

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Fault-tolerant preparation of ancilla Θ

- Ancilla state Θ
  - $|0\rangle$: Eigenstate of *Z*
  - $|\Theta\rangle$: Eigenstate of $THZHT^\dagger = e^{-i\pi/4}SX$
    Eigenvalue +1
- Fault-tolerant measurement of $e^{-i\pi/4}SX$
  - If measurement result = +1
    - Ancilla state Θ is prepared
  - If measurement result = –1
    - Discard and repeat; OR
    - Apply fault-tolerant *Z*

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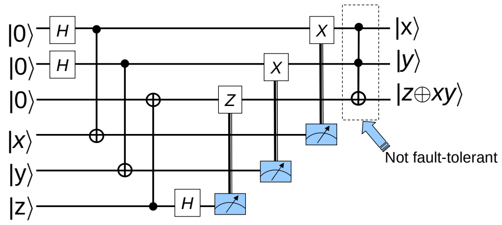
Fault-tolerant Toffoli gate (1)

- Swap $|xyz\rangle$ with known state $|000\rangle$



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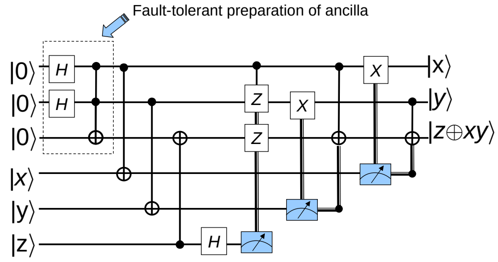
Fault-tolerant Toffoli gate (2)

- Use commutation rules to move Toffoli gate to the left



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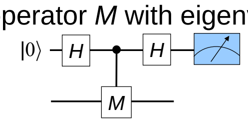
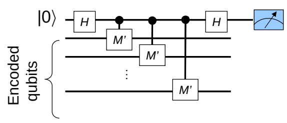
Measurement of operators (not fault-tolerant)

- Circuit to measure a single qubit operator *M* with eigenvalue $m = \pm 1$



$$|0\rangle|\psi\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|\psi\rangle$$
$$\xrightarrow{\text{C-}M} \frac{1}{\sqrt{2}}\left(|0\rangle + m|1\rangle\right)|\psi\rangle$$
$$\xrightarrow{H} \begin{cases} |0\rangle|\psi\rangle & m = +1 \\ |0\rangle|\psi\rangle & m = -1 \end{cases}$$

- Operator *M* on encoded qubits
  - *M*: Bitwise application of *M'*



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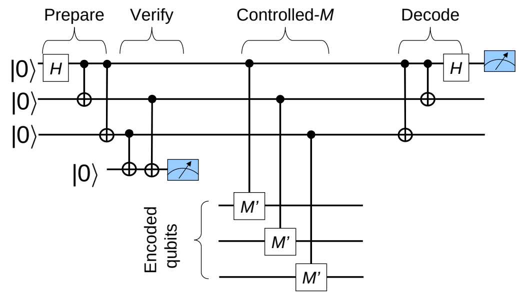
Fault-tolerant measurement of operators



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Accuracy threshold

Quantum error code to correct many errors

- Quantum error correcting code [[*n, k, 2t+1*]]
  - Uncorrectable if *t*+1 independent errors (each with probability ε) occur before error correction
  - Probability $\varepsilon^{t+1} \ll \varepsilon$; but
- Longer codes require longer steps in syndrome measurement
  - Uncorrectable errors accumulate before they can be corrected!

→ Concatenated code

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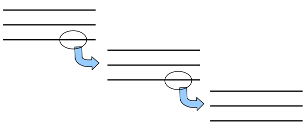
Concatenated code

- The 7-qubit (Steane) code
  - Probability of failure ε^2
- Concatenated once: 7^2 blocks
  - Probability of failure in the sub-block: ε^2
  - Total probability of failure: $(\varepsilon^2)^2$
- *L* levels of concatenation: 7^L blocks
  - Total probability of failure: $(\varepsilon^2)^L$



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Estimation of accuracy threshold

- Probability of failure:
  - At each level of concatenation, a block of 7 qubits fails if two or more sub-blocks contain errors:

$$p_{L+1} \approx \binom{7}{2} p_L^{\ 2} + \cdots = 21 p_L^{\ 2} + \cdots$$

  - Threshold value: p_0 =1/21
- Relate gate errors (ε_{gate}) and storage error ($\varepsilon_{storage}$) to p_0
  - Assuming error correction after every CNOT gate
  - → Threshold error rates
    - $\varepsilon_{gate} \sim 6 \cdot 10^{-4}$
    - $\varepsilon_{storage} \sim 6 \cdot 10^{-4}$

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Shor's algorithm?

- Factorize a 130-digit (432-bit) number
  - A few months by a classical computer
  - Shor's algorithm
    - Requirements
      - 5×432 qubits
      - $38 \times (432)^3 \sim 3 \times 10^9$ Toffoli gates
      - Gate error < 10^{-9} per Toffoli gate
      - Storage error < 10^{-12} per gate operation time
    - Can be achieved by
      - 3 levels of concatenation of 7-qubit Steane code
      - Individual error rates $\varepsilon_{gate} \sim \varepsilon_{storage} \sim 10^{-6}$
    - Total number of qubits $\sim 10^6$
      - including additional ancillary qubits to implement gates fault-tolerantly

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