Stabilizer codes

What is the stabilizer formalism?

- Represent a quantum state by the operators stabilizing it.
- Example: EPR state
  - State (ordinary) representation

    $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$
  - Stabilizer: X_1X_2 and Z_1Z_2

    $X_1X_2|\psi\rangle = |\psi\rangle, \quad Z_1Z_2|\psi\rangle = |\psi\rangle$
    - Uniquely determined.

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Pauli group on n qubits G_n

- Single qubit error $E = e_0 I + e_1 X + e_2 Z + e_3 XZ$
  - → Useful to express errors in terms of Pauli matrix
- Pauli group
  - on one qubit
    - $G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$
      - $XY = iZ$, $YX = -iZ$
      - $X(iX) = iI$
  - on two qubits
    - $G_2 = \{\pm II, \pm iII, \pm IX, \pm iIX, \pm XI, \pm iXI, \cdots\}$
  - on n qubits
    - G_n

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Properties of Pauli group G_n

1. Each $M \in G_n$ is unitary

   $M^{-1} = M^\dagger$
2. For each $M \in G_n$, $M^2 = \pm I \equiv \pm I^{\otimes n}$
3. M is Hermitian or anti-Hermitian
   - If $M^2 = I$, then M is Hermitian ($M = M^\dagger$)
   - If $M^2 = -I$, then M is anti-Hermitian ($M = -M^\dagger$)
4. Any two elements $M, N \in G_n$ either commute or anti-commute: $MN = \pm NM$

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Definition of stabilizers

- Stabilizer S of the code:

  $|\psi\rangle \in V_S$ iff $\boldsymbol{M}|\psi\rangle = |\psi\rangle$ for all $\boldsymbol{M} \in S$
  - S: subset of G_n (Pauli group on n qubits)
  - Vector space V_S stabilized by S
    - = simultaneous eigenspace with eigenvalue +1 of all elements of S
- Example: EPR state
  - $S = \{X_1X_2, Z_2Z_2\}$ (Subset of G_2)
  - $V_S = \{ \frac{|00\rangle + |11\rangle}{\sqrt{2}} \}$

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Properties of stabilizers

- If $-I \in S$, $V_S = \{|0\rangle\}$
  - Example:
    - $S = \{\pm I, \pm X\}$ (Subgroup of G_1)
    - Vector stabilized by $(-I)$

    $(-I)|\psi\rangle = |\psi\rangle \Rightarrow |\psi\rangle = 0$
- If $[M_1, M_2] \neq 0$ ($M_1, M_2 \in S$), $V_S = \{|0\rangle\}$
  - Example:
    - $\{X, Y, \cdots\}$: Subgroup of G_1
    - Vector stabilized by X and Y

    $\underbrace{(XY - YX)}_{\neq 0}|\psi\rangle = |\psi\rangle \Rightarrow |\psi\rangle = 0$
- → Stabilizer S for non-trivial vector space
  - $-I \notin S$
  - All elements commute

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Properties of stabilizers II

- Number of encoded qubits
  - Use n qubits ($n \geq k$)
  - Generators of stabilizer $\{M_i\}$ ($i = 1, \cdots, n-k$)
    - Those elements are independent (no one can be expressed as a product of others)
    - Each element can be expressed as a product of elements of $\{M_i\}$.
  - → Code subspace V_s has dimension 2^k
  - ≡ Encode k qubits

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Number of encoded qubits

- $M \in S$ must satisfy $M^2 = I$
  - If $M^2 = -I$, M cannot have the eigenvalue +1
- For each $M \neq \pm I$ in G_n, the eigenvalues +1 and –1 have equal degeneracy
  - For each $M \neq \pm I$, there is an $N \in G_n$ that anti-commutes with M; $MN = -NM$
    - $M|\psi\rangle = |\psi\rangle$ [Eigenvalue +1]
    - $M(N|\psi\rangle) = -N|\psi\rangle$ [Eigenvalue –1]
    - ½ $(2^n) = 2^{n-1}$ mutually orthogonal states satisfy $M|\psi\rangle = |\psi\rangle$
- For $\{M_i\}$ ($i = 1, \cdots, n-k$)
  - $(½)^{n-k}$ $(2^n) = 2^k$ mutually orthogonal states satisfy $M_i|\psi\rangle = |\psi\rangle$

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Detection and correction of errors in stabilizer formalism

- Stabilizer generators $\{M_i\}$ ($i = 1, \cdots, n-k$) as check operators
  - Collective measurement of the generators
    - No error: $M_j = 1$ for each generator
    - Error: $M_j = -1$ for some j
  - → Detect and correct the error
- Error syndrome
  - Error operator $E_a \in$ Pauli group either commutes or anti-commutes with M_j
    - E_a commutes with M_j :
      $M_j E_a|\psi\rangle = E_a M_j|\psi\rangle = E_a|\psi\rangle$
    - E_a anti-commutes with M_j :
      $M_j E_a|\psi\rangle = -E_a M_j|\psi\rangle = -E_a|\psi\rangle$

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Condition for error recovery in stabilizer formalism

- Condition for error recovery
  $$\langle i|E_a^\dagger E_b|j\rangle = C_{ab}\delta_{i,j} \quad (*)$$
  - $E_a, E_b \in \Xi$
  - $|i\rangle, |j\rangle$: encoded states (normalized)
  - C_{ab}: independent of $|i\rangle, |j\rangle$
- Condition $(*)$ is satisfied when
  1. $E_a^\dagger E_b \in S$; or
  2. There is an $M \in S$ that anti-commutes with $E_a^\dagger E_b$

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When error correction fails

- If $E_a^\dagger E_b \notin S$ commutes with stabilizer
  - preserves the code subspace
  - acts as logic operations on the encoded qubits
- A stabilizer code with distance d
  - Each error $E \in G_n$ of weight $< d$
    - lies in the stabilizer; or
    - anti-commutes with some generators of the stabilizer

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Error correcting code in stabilizer formalism

- Code subspace = Vector space V_S stabilized by S
- Example: 3-qubit bit flip code
  - $S = \{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$
  - Subspaces stabilized by
    - Z_1Z_2 : $|000\rangle, |001\rangle, |110\rangle, |111\rangle$
    - Z_2Z_3 : $|000\rangle, |011\rangle, |100\rangle, |111\rangle$
    - Z_1Z_3 : $|000\rangle, |010\rangle, |101\rangle, |111\rangle$
  - → $V_S = \{|000\rangle, |111\rangle\}$ = Code subspace
  - $S = \langle Z_1Z_2, Z_2Z_3\rangle$: using generators
    - at most log|G| generators
    - $I = (Z_1Z_2)^2$, $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$
    - Determine subspaces stabilized by Z_1Z_2 and Z_2Z_3 only

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Example: 3-qubit bit/phase flip code

- 3-qubit bit flip code
  - Stabilizer S
    $\langle Z_1Z_2, Z_2Z_3\rangle$
  - Code subspace
    $\{|000\rangle, |111\rangle\}$
  - Bit flip error detection
    - $Z_1Z_2 = -1$ (if X_1 or X_2 error occurs)
    - $Z_2Z_3 = -1$ (if X_2 or X_3 error occurs)
- 3-qubit phase flip code
  - Stabilizer S
    $\langle X_1X_2, X_2X_3\rangle$
  - Code subspace
    $\{|{+}{+}{+}\rangle, |{-}{-}{-}\rangle\}$
  - Phase flip error detection
    - $X_1X_2 = -1$ (if Z_1 or Z_2 error occurs)
    - $X_2X_3 = -1$ (if Z_2 or Z_3 error occurs)

Syndrome measurement = generator measurement

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Symplectic notation

- Any element $M \in G_n$
  - can be expressed as a product: $M = Z_M \cdot X_M$ (up to phase)
    - Z_M: tensor product of Z's
    - X_M: tensor product of Xs
  - → $(\alpha|\beta) \equiv Z(\alpha)X(\beta) = \otimes_{i=1}^{n} Z^{\alpha_i} \otimes_{i=1}^{n} X^{\beta_i}$
- Characterization of the code
  - $(n\text{-}k)\times 2n$ matrix: $H = (H_Z|H_X)$
    - n-k generators
    - n total qubits

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Example: 9-qubit (Shor) code [[9,1,3]]

- Stabilizer generators
  - $Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9$
  - → Detect bit flip errors
  - $X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9$
  - → Detect phase flip errors

$$\left[\begin{array}{ccccccccc|ccccccccc}
1&1&0&&&&&&&&&&&&&&&\\
0&1&1&&&&&&&&&&&&&&&\\
&&&1&1&0&&&&&&&&&&&&\\
&&&0&1&1&&&&&&&&&&&&\\
&&&&&&1&1&0&&&&&&&&&\\
&&&&&&0&1&1&&&&&&&&&\\
\hline
&&&&&&&&&1&1&1&1&1&1&0&0&0\\
&&&&&&&&&0&0&0&1&1&1&1&1&1
\end{array}\right] = \begin{bmatrix} H_Z & 0 \\ 0 & H_X \end{bmatrix}$$

- Code subspace

$$\left\{ \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}, \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}} \right\}$$

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Example: 7-qubit (Steane) code [[7,1,3]]

$$H_{\text{Ham}} \equiv \begin{bmatrix} 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1 \end{bmatrix}$$

- Stabilizer generators
  - $Z_4Z_5Z_6Z_7, Z_2Z_3Z_6Z_7, Z_1Z_3Z_5Z_7$
  - → Detect bit flip errors
  - $X_4X_5X_6X_7, X_2X_3X_6X_7, X_1X_3X_5X_7$
  - → Detect phase flip errors

$$\begin{bmatrix} H_{\text{Ham}} & 0 \\ 0 & H_{\text{Ham}} \end{bmatrix}$$

- Code subspace

$$\left\{ \frac{1}{\sqrt{8}} \sum_{\text{even } w \in \text{Hamming}} w, \frac{1}{\sqrt{8}} \sum_{\text{odd } w \in \text{Hamming}} w \right\}$$

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Example: CSS code / more general CSS code

- CSS code
  - Classical code $C \subset C^{\perp}$
    - Parity check matrix H for C
    - → $[[n, 2k\text{-}n, d]]$ quantum code
  - Stabilizer generators $\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$
- More general CSS code
  - Stabilizer generators $\begin{bmatrix} H_Z & 0 \\ 0 & H_X \end{bmatrix}$

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Logic operations on encoded qubits

- $2k$ logic operations:
  - $(\alpha|\beta)$: $2n$-dimensional space
  - n–k independent generators
  - $2n$–$(n$–$k)$=n+k vectors commute with the generators
  - Among them, n–k vectors are generators
  - → Remaining $2k$ vectors
    - commute with stabilizer
    - act non-trivially on the k encoded qubits (*logic operations*)

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Pauli operators on encoded qubits

- $2k$ logic operations:
  - Pauli operators Z and X on encoded qubits (i = 1, 2, ⋯, k) $\overline{Z}_i, \overline{X}_i$
  - Z operators:
    - Choose k operators $\overline{Z}_1, \overline{Z}_2, \cdots, \overline{Z}_k$ such that n vectors (n–k generators + k operators) mutually commute
    - Logical state $\left|\overline{z_1}, \overline{z_2}, \cdots, \overline{z_k}\right\rangle$
  - X operators:
    - Commute with stabilizer $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_k$
    - Anti-commute with $\overline{Z}_j$

$$\overline{Z}_i \overline{X}_j = (-1)^{\delta_{ij}} \overline{X}_j \overline{Z}_i$$

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Examples

- The 9-qubit (Shor) code
  - Pauli operators $\overline{Z} = X_1X_2X_3,\ \overline{X} = Z_1Z_4Z_7$
- The 7-qubit (Steane) code
  - Pauli operators $\overline{X} = X_1X_2X_3,\ \overline{Z} = Z_1Z_2Z_3$
  - Stabilizer generators
    - $M_{Z1}=Z_4Z_5Z_6Z_7$, $M_{Z2}=Z_2Z_3Z_6Z_7$, $M_{Z3}=Z_1Z_3Z_5Z_7$
    - $M_{X_1}=X_4X_5X_6X_7$, $M_{X2}=X_2X_3X_6X_7$, $M_{X3}=X_1X_3X_5X_7$
  - Encoded states

$$\left|\overline{0}\right\rangle = (1+M_{X1})(1+M_{X2})(1+M_{X3})\left|00\cdots0\right\rangle$$
$$\left|\overline{1}\right\rangle = (1+M_{X1})(1+M_{X2})(1+M_{X3})\left|11\cdots1\right\rangle$$

$M_{Zi}\left|00\cdots0\right\rangle = \left|00\cdots0\right\rangle,\ \overline{Z}\left|00\cdots0\right\rangle = \left|00\cdots0\right\rangle$

$M_{Zi}\left|11\cdots1\right\rangle = \left|11\cdots1\right\rangle,\ \overline{Z}\left|11\cdots1\right\rangle = -\left|11\cdots1\right\rangle$

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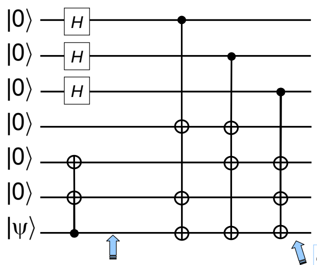
Circuit for encoding



$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

- $M_{X_1}=X_1X_4X_6X_7$, $M_{X2}=X_2X_4X_5X_7$, $M_{X3}=X_3X_4X_5X_6$

$a\left|0000000\right\rangle + b\left|0000111\right\rangle$

$a(1+M_{X1})(1+M_{X2})(1+M_{X3})\left|0000000\right\rangle + b(1+M_{X1})(1+M_{X2})(1+M_{X3})\left|1110000\right\rangle$

$$\left|0000111\right\rangle = X_1X_2X_3X_4\left|1111111\right\rangle = M_{X1}M_{X2}M_{X3}\left|1111111\right\rangle$$

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