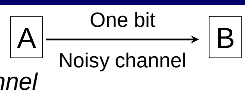
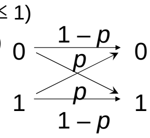
Quantum error correction

Classical error correction

- Example:



  - *Binary symmetric channel*
    - Noise in the channel
    = flip the bit with probability *p* $(0 \le p \le 1)$
  - Three-bit code (*Majority voting*)



    1. Encoding (duplication)
       0 → 000 (logical 0)
       1 → 111 (logical 1)
    2. Error detection (measurement)
       Output 001 → 000
    - Fails if two or more of the bits are flipped
      $3p^2(1-p)+p^3 = p_e$: probability of error ≤ *p* (if $p \le ½$)

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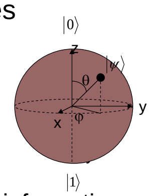
Quantum error correction

- Differences from classical cases
  - No-cloning theorem:
    - Impossible to make copies
    → No *duplication*
  - Continuous errors:
    - Errors are not limited to 0↔1



  - *Measurement* destroys quantum information

$$|\psi\rangle = a|0\rangle + b|1\rangle \xrightarrow{\text{Measurement}} \begin{cases} |0\rangle & \text{with probability } |a^2| \\ |1\rangle & \text{with probability } |b^2| \end{cases}$$

  → Makes recovery impossible?
- Quantum error correction still works!

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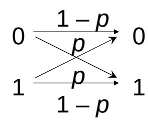
Example (1): 3-qubit bit flip code

- Bit flip channel
  - $\psi\rangle \to X|\psi\rangle$ with probability *p*

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{(Pauli matrix) : Bit flip operator}$$

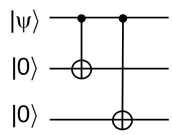
$$a|0\rangle + b|1\rangle \to \begin{cases} a|0\rangle + b|1\rangle & \text{with probability } 1-p \\ a|1\rangle + b|0\rangle & \text{with probability } p \end{cases}$$



- Assumption:
  - At most one error occurs (small *p*)
- Encoding

$$|\psi\rangle = a|0\rangle + b|1\rangle \to a|000\rangle + b|111\rangle$$

$|0_L\rangle = |000\rangle$
$|1_L\rangle = |111\rangle$



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Error detection & recovery

- "Syndrome" measurements
  1. $Z_1Z_2 (= Z \otimes Z \otimes I) = \begin{cases} +1 & \text{if qubit 1} = \text{qubit 2} \\ -1 & \text{if qubit 1} \ne \text{qubit 2} \end{cases}$
     (Bit flip on one of the bits)
  2. $Z_2Z_3 (= Z \otimes Z \otimes I) = \begin{cases} +1 & \text{if qubit 2} = \text{qubit 3} \\ -1 & \text{if qubit 2} \ne \text{qubit 3} \end{cases}$
     (Bit flip on one of the bits)
  → Determine which qubit is flipped.
- Recovery
  - Apply X_1, X_2, or X_3 depending on the "syndrome"
- Examples
  - |000⟩ (No error): $Z_1Z_2 = +1$, $Z_2Z_3 = +1$ → Do nothing
  - |010⟩ (Error on #2): $Z_1Z_2 = -1$, $Z_2Z_3 = -1$ → Apply X_2 → |000⟩
  - |110⟩ (Error on #3): $Z_1Z_2 = +1$, $Z_2Z_3 = -1$ → Apply X_3 → |111⟩

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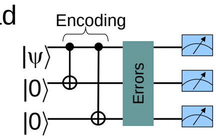
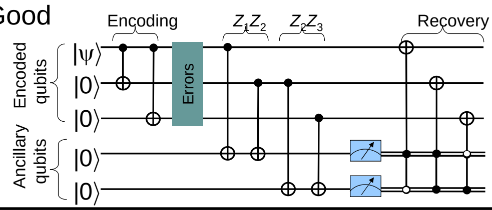
Circuit for syndrome measurements

- Bad



Measurement destroys quantum information!

- Good



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When error correction fails ≡ Logic operation

- When two errors occur:
  - Example
    - Bit-flip errors on #1 & #2: **|000⟩** → |110⟩
    - Syndrome: Z_1Z_2 = +1, Z_2Z_3 = –1 → (Error on #3)
    - Recovery: Apply X_3 → **|111⟩**
    - → Logic operation $\overline{X} \equiv X_1X_2X_3$
  - Error correction fails. Equivalent to logic operation on encoded qubits.
- Probability of failure
  - $3p^2(1–p) + p^3 < p$ (when $p < ½$)

2 bit-flips 3 bit-flips No error correction

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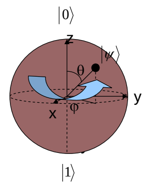
Example (2): 3-qubit phase flip code

- Phase flip error
  - Relative phase between |0⟩ and |1⟩ is flipped with probability p
  - $|\psi\rangle \rightarrow Z|\psi\rangle$ with probability p

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \text{Phase flip operator}$$

$$a|0\rangle + b|1\rangle \rightarrow \begin{cases} a|0\rangle + b|1\rangle & \text{with probability } 1-p \\ a|0\rangle - b|1\rangle & \text{with probability } p \end{cases}$$



  - No classical equivalent
    - No phases in classical channels

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Phase flip error ≡ Bit flip error

- Convert phase flip error into bit flip error
  - Consider the basis

$$|+\rangle \equiv H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2},\ |-\rangle \equiv H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

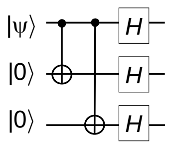
  - Phase flip error (Z):

$$Z|+\rangle = (|0\rangle - |1\rangle)/\sqrt{2} = |-\rangle,\ Z|-\rangle = (|0\rangle + |1\rangle)/\sqrt{2} = |+\rangle$$

  - Bit flip error in | ± ⟩ basis!
- Encoding

$$|\psi\rangle = a|0\rangle + b|1\rangle \rightarrow a|+++\rangle + b|---\rangle$$

$|0_L\rangle$ = |+++⟩
$|1_L\rangle$ = |- - -⟩



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Error detection & recovery

- Syndrome measurements
  1. $X_1X_2 (= X \otimes X \otimes I) = \begin{cases} +1 & \text{if qubit 1} = \text{qubit 2} \\ -1 & \text{if qubit 1} \neq \text{qubit 2} \end{cases}$ (Phase flip on one of the bits)
  2. $X_2X_3 (= X \otimes X \otimes I) = \begin{cases} +1 & \text{if qubit 2} = \text{qubit 3} \\ -1 & \text{if qubit 2} \neq \text{qubit 3} \end{cases}$ (Phase flip on one of the bits)
  - → Determine which qubit is flipped.
- Recovery
  - Apply Z_1, Z_2, or Z_3 depending on the “syndrome”
- Examples
  - |+++⟩ (No error): X_1X_2 = +1, X_2X_3 = +1 → Do nothing
  - |+–+⟩ (Error on #2): X_1X_2 = –1, X_2X_3 = –1 → Apply Z_2 → |+++⟩
  - |- -+⟩ (Error on #3): X_1X_2 = +1, X_2X_3 = –1 → Apply Z_3 → |- - -⟩

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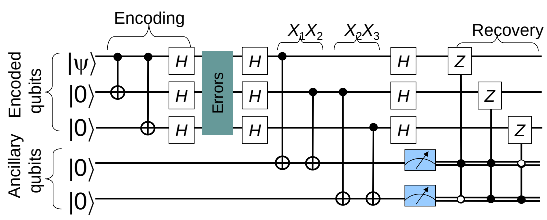
Circuit for syndrome measurements



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When error correction fails ≡ Logic operation

- When two errors occur:
  - Example
    - Bit-flip errors on #1 & #2: **|+++⟩** → | – – +⟩
    - Syndrome: X_1X_2 = +1, X_2X_3 = –1 → (Error on #3)
    - Recovery: Apply Z_3 → **| – – –⟩**
    - → Logic operation $\overline{Z} \equiv Z_1Z_2Z_3$
  - Error correction fails. Equivalent to logic operation on encoded qubits.
- Probability of failure
  - $3p^2(1–p) + p^3 < p$ (when $p < ½$)

2 bit-flips 3 bit-flips No error correction

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Arbitrary errors

- Single qubit errors are continuous
  - 3-qubit bit flip code → only X errors
  - 3-qubit phase flip code → only Z errors
- Single qubit error E = 2×2 Unitary matrix (Trace preserving)

$$E = e_0 I + e_1 X + e_2 Z + e_3 XZ$$

  - Before an error: $|\psi\rangle = a|0_L\rangle + b|1_L\rangle$
  - After: $E|\psi\rangle$ = linear combination of $|\psi\rangle$, $Z_1|\psi\rangle$, $X_1|\psi\rangle$, and $X_1Z_1|\psi\rangle$
- → Only a discrete subset of those errors (X, Z, or XZ) can correct *arbitrary errors!*

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Example (3): Shor code

- Protect against an *arbitrary* error on a single qubit:
  - *Concatenation*: using a hierarchy of levels
- Encoding
  - Encode the qubit using three-qubit phase flip code (Z error) $|0\rangle \to |+++\rangle, |1\rangle \to |---\rangle,$
  - Encode each of the qubits using the three-qubit bit flip code (X error)

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \to \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \to \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

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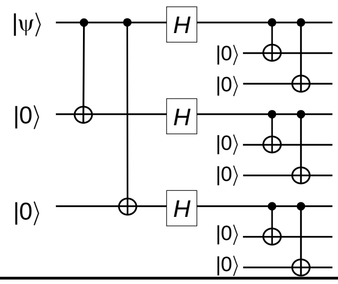
Circuit for encoding

$$|0\rangle \to \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \to \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



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Error detection & recovery

- Bit flip error on qubit 1
  - Syndrome measurement
    - Z_1Z_2 → –1
    - Z_2Z_3 → +1
    - ⇒ Bit flip error on qubit 1
  - Recovery: Apply X_1
- Phase flip error on qubit 1

  $|000\rangle + |111\rangle \to |000\rangle - |111\rangle$
  - Syndrome measurement
    - $X_1X_2X_3X_4X_5X_6$ → –1
    - $X_4X_5X_6X_7X_8X_9$ → +1
    - ⇒ Phase flip error on qubit 1, 2 or 3
  - Recovery: Apply $Z_1Z_2Z_3$ (Z_1, Z_2 or, Z_3)

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When error correction fails

- Two bit-flip errors in one cluster
  - Example
    - Bit-flip errors on #1 & #2: X_1X_2
    - Syndrome: Z_1Z_2 = +1, Z_2Z_3 = –1 → (Error on #3)
    - Recovery: Apply X_3
    - → Phase flip to the cluster $X_1X_2X_3(|000\rangle \pm |111\rangle) \to |000\rangle \mp |111\rangle$
- Two phase-flip errors in two clusters
  - Example
    - Phase-flip errors on #1 & #4: Z_1Z_4
    - Syndrome: $X_1X_2X_3X_4X_5X_6$ = +1, $X_4X_5X_6X_7X_8X_8$ = –1 → (Error on the 3rd cluster)
    - Recovery: Apply Z_7
    - → Bit flip $Z_1Z_2Z_3(\alpha|0_L\rangle + \beta|1_L\rangle) \to \alpha|1_L\rangle + \beta|0_L\rangle$

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Quantum error correction without measurement

- Error correction with measurement
  1. Error detection by measurement
  2. Recovery by conditional operation
  - Measurement is difficult
- Error correction without measurement
  - Measurement operator: M_i
  - Conditional operator: U_i
  - Refresh ancillary qubit $|0\rangle \to |i\rangle$ (syndrome)
  - Unitary operator U
  - → Detect & correct errors

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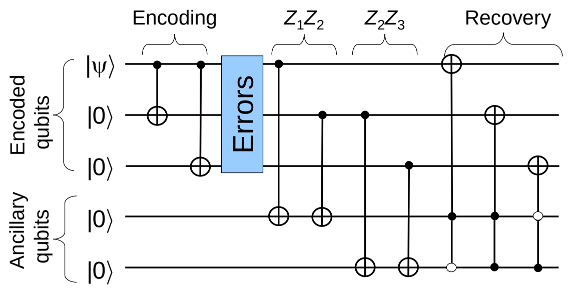
Circuit for QEC without measurement

- 3-qubit bit flip code



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Quantum Hamming bound

- How many qubits *n* required to correct *t* or less errors?
  - *k* qubits encoded in *n* qubits
  - Non-degenerate codes
  - When *j* errors occur ($j \leq t$)
    - $\binom{n}{j}$ possible locations of errors
    - Three possible errors *X*, *Z*, *XZ* on each qubit
    - → Total number of errors $\sum_{j=0}^{t} \binom{n}{j} 3^j$ ⇨ $\sum_{j=0}^{t} \binom{n}{j} 3^j 2^k \leq 2^n$
- $k = 1 (t = 1)$

$$\sum_{j=0}^{1} \binom{n}{j} 3^j 2^1 = (1+3n)2 \leq 2^n \Rightarrow n \geq 5$$

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